

# Some extensions of the uncertainty principle

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## Abstract

We study the formulation of the uncertainty principle in quantum mechanics in terms of entropic inequalities, extending results recently derived by Bialynicki-Birula [1] and Zozor *et al.* [2]. Those inequalities can be considered as generalizations of the Heisenberg uncertainty principle, since they measure the mutual uncertainty of a wave function and its Fourier transform through their associated Rényi entropies with conjugated indices. We consider here the general case where the entropic indices are not conjugated, in both cases where the state space is discrete and continuous: we discuss the existence of an uncertainty inequality depending on the location of the entropic indices  $\alpha$  and  $\beta$  in the plane  $(\alpha, \beta)$ . Our results explain and extend a recent study by Luis [3], where states with quantum fluctuations below the Gaussian case are discussed at the single point  $(2, 2)$ .

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## 1 Introduction

The Uncertainty Principle (UP) is such a fundamental concept that it focuses great attention not only in quantum physics but even in other areas (*e.g.* signal or image processing). In quantum mechanics terms, the UP establishes the

existence of an irreducible lower bound for the uncertainty in the result of a simultaneous measurement of non-commuting observables. An alternative expression is that the precision with which incompatible physical observables can be prepared is limited by an upper bound. Quantitatively, the UP can be given by a relation of the form  $U(A, B; \Psi) \geq \mathcal{B}(A, B)$ , where  $U$  measures the uncertainty in the simultaneous preparation or measurement of the pair of operators  $A$  and  $B$  when the quantum system is in state  $\Psi$ , while  $\mathcal{B}$  is a state-independent bound. The quantity  $U$  should take a fixed minimum value if and only if  $\Psi$  is a common eigenstate of both operators. It is interesting to note, as remarked by Deutsch [4], that the quantitative formulation of the UP through generalizations of Heisenberg's inequality to an arbitrary pair of non-commuting observables (other than position and momentum) may present some drawbacks, as the expectation value of the commutator between both operators (unless it is a *c*-number) does depend on the current state of the system. Many authors have contributed to the formulation of alternative quantitative expressions for the UP, among which is the use of *entropic* measures for the uncertainty, inspired by information theory (see, for instance, [5] and references therein). We address here the search for lower bounds of uncertainty relations given in terms of generalized entropies of the Rényi form.

The paper is organized as follows. In section 2, we give some brief recalls on Fourier transform in the context of quantum physics, and on Rényi entropy, which can be viewed as an extension of Shannon entropy. Also, we summarize properties of entropic uncertainty inequalities for which the Rényi entropies have conjugated indices. Section 3 is the core of this work : its aim is to extend the usual entropic uncertainty inequalities using Rényi entropies with arbitrary indices. We show in particular that an uncertainty principle in such a form does not always exist for arbitrary pairs of indices. In section 4, we present some conclusions, while detailed proofs of our main results are given in the appendices at the end of the paper.

## 2 Previous results on entropic uncertainty relations

We consider a  $d$ -dimensional operator  $A$  and we assume that the state of a system is described by the wavefunction  $\Psi$ . Let us now denote  $\hat{\Psi}$  the Fourier transform of  $\Psi$ , assumed to describe the state of operator  $\tilde{A}$ . In the following, we will refer to  $A$  and  $\tilde{A}$  as conjugate operators, *i.e.* operators having wavefunctions that are linked by a Fourier transformation. As an example, we may consider the position and momentum of a particle, or the position and its angular momentum.

The definition of Fourier transform depends on the state space considered: if the state space is discrete taking values on the alphabet  $\mathcal{A}^d = \{\dots, 0, 1, \dots\}^d$ ,

finite or not, the Fourier transform of  $\Psi$  takes the form

$$\hat{\Psi}(\mathbf{x}) = (2\pi)^{-\frac{d}{2}} \sum_{\mathbf{a} \in \mathcal{A}^d} \Psi(\mathbf{a}) e^{-i\mathbf{a}^t \mathbf{x}}, \quad (1)$$

where  $\mathbf{x} \in [0; 2\pi)^d$ . In the particular case of a discrete and finite alphabet of size  $n$ ,  $\mathcal{A} = \{0, \dots, n-1\}$ , one can also consider a discrete Fourier transform

$$\hat{\Psi}_{\text{df}}(\mathbf{k}) = n^{-\frac{d}{2}} \sum_{\mathbf{a} \in \mathcal{A}^d} \Psi(\mathbf{a}) e^{-i2\pi\mathbf{a}^t \mathbf{k}/n}, \quad (2)$$

where  $\mathbf{k} \in \mathcal{A}^d$ . When dealing with periodic quantities such as angular momentum, Eq. (1) is to be considered [1,6]; Eq. (2) describes a system where the state and its conjugate state are both finite and discrete.

Finally, in the continuous case, the Fourier transform takes the form

$$\hat{\Psi}(\mathbf{x}) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \Psi(\mathbf{u}) e^{-i\mathbf{u}^t \mathbf{x}} d\mathbf{u}. \quad (3)$$

Let us consider now the Rényi  $\lambda$ -entropy associated to the operator  $A$  when the physical system is in a state described by the wavefunction  $\Psi$ ,

$$H_\lambda(A) = \frac{2\lambda}{1-\lambda} \ln \|\Psi\|_{2\lambda}, \quad (4)$$

for any real positive  $\lambda \neq 1$ , where  $\|\cdot\|_p$  denotes the standard  $p$ -norm of a function:  $\|\Psi\|_p = (\int_{\mathbb{R}^d} |\Psi(\mathbf{u})|^p d\mathbf{u})^{1/p}$ . When  $\lambda \rightarrow 1$ , the Rényi entropy  $H_\lambda$  converges to the usual Shannon entropy

$$H_1(A) = - \int_{\mathbb{R}^d} |\Psi(\mathbf{u})|^2 \ln |\Psi(\mathbf{u})|^2 d\mathbf{u}. \quad (5)$$

When  $\lambda = 0$  one gets  $H_0(A) = \ln \mu(\{x, \Psi(x) \neq 0\})$  where  $\mu(\cdot)$  is the Lebesgue measure (*i.e.*  $H_0$  equals the logarithm of the volume of the support of  $\Psi$ ). At the opposite, when  $\lambda \rightarrow +\infty$ , one has  $H_\infty = -\ln \|\Psi\|_\infty = -\ln \sup_{x \in \mathbb{R}^d} |\Psi(x)|$ .

The extension to a discrete state space is straightforward by replacing the integral by a discrete sum. Equivalently, the Rényi  $\lambda$ -entropy power associated

to  $A$  can be defined as<sup>1</sup>

$$N_\lambda(A) = \exp\left(\frac{1}{d} H_\lambda(A)\right). \quad (6)$$

In the one-dimensional finite discrete case, this entropy power is the inverse of the *certainty* measure  $M_{\lambda-1}$  defined by Maassen and Uffink in Ref. [8].

Among the properties of the one-parameter family of Rényi entropies, let us mention that for arbitrary fixed  $A$  (or  $\Psi$ ),  $H_\lambda$  is non increasing versus  $\lambda$ , and hence  $N_\lambda$  is also non increasing against  $\lambda$  [9, th. 192]. This property can easily be shown by computing the derivative against  $\lambda$ , the derivative of the factor of  $1/(1-\lambda)^2$ , and using the Cauchy-Schwartz inequality. How  $N_\lambda$  decreases against  $\lambda$  is then intimately linked to  $\Psi$ : in the particular case of a constant  $\Psi$  (on a bounded support, in order to ensure wavefunction normalization),  $N_\lambda$  is constant. This is illustrated in figure 1 where the entropy power  $N_\lambda(A)$  is plotted versus  $\lambda$  for various observables  $A$  with corresponding wavefunctions  $\Psi$ . This figure illustrates also the fact that the maximal entropy power is not given by the same  $\Psi$  for any  $\lambda$ : under covariance matrix constraint,  $N_\lambda$  is maximal in the Gaussian case for  $\lambda = 1$ , in the Student- $t$  case with  $\nu$  degrees of freedom for  $\lambda = 1 - \frac{2}{d+\nu}$  or in the Student- $r$  case with  $\nu$  degree of freedom for  $\lambda = 1 + \frac{2}{\nu-d}$  [10,11].

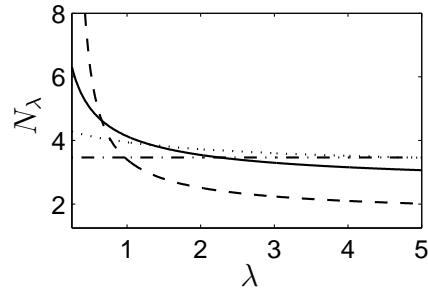


Fig. 1. Behavior of  $N_\lambda$  versus  $\lambda$  for various wavefunctions (identity covariance matrix,  $d = 1$ ) : Gaussian case (solid line), Student- $t$  case  $\Psi(\mathbf{x}) \propto (1 + \mathbf{x}^t \mathbf{x}/(\nu - 2))^{-\frac{d+\nu}{4}}$  for  $\nu = 3$  (dashed line), Student- $r$  case  $\Psi(\mathbf{x}) \propto (1 - \mathbf{x}^t \mathbf{x}/(\nu + 2))^{\frac{\nu-d}{4}}$  for  $\nu = 3$  (dotted line) and Uniform on the sphere, i.e. Student- $r$  with  $\nu = d$  (dash-dotted line).

Uncertainty relations involving  $A$  and  $\tilde{A}$  that make use of certain combinations of Rényi generalized entropies have already been established in Ref. [1,2] in the continuous-continuous, discrete-continuous (periodic) and discrete-discrete cases. In terms of the Rényi  $\lambda$ -entropy power, these uncertainty relations read

$$N_{\frac{p}{2}}(A) N_{\frac{q}{2}}(\tilde{A}) \geq C_{p,q}, \quad (7)$$

<sup>1</sup> In [7, p. 499] the Rényi entropy power is defined as  $\exp(2H_\lambda/d)$ : our definition does not affect the content of the paper and is in concordance with almost all cited papers. In the Shannon case, the definition  $\frac{1}{2\pi e} \exp(2H_\lambda/d)$  can also be found: with this last definition, the entropy power and the variance coincide in the Gaussian situation.

where the entropic parameters  $\frac{p}{2}$  and  $\frac{q}{2}$  have been chosen *conjugated*<sup>2</sup>, *i.e.*  $\frac{1}{p} + \frac{1}{q} = 1$ , and with  $p \geq 1$  (infinite  $q$  when  $p = 1$ ). The constant lower bound is expressed as

$$C_{p,q} = \begin{cases} n & \text{in the discrete-discrete case} \\ 2\pi p^{\frac{1}{p-2}} q^{\frac{1}{q-2}} & \text{in the continuous-continuous case} \\ 2\pi & \text{in the discrete-continuous case.} \end{cases} \quad (8)$$

In the continuous-continuous case, we extend by continuity  $C_{p,q}$  as  $C_{1,+\infty} = 2\pi$  and  $C_{2,2} = e\pi$ . In the discrete case (finite or not), this kind of uncertainty relation is usually exhibited for one-dimensional quantities, but it extends trivially to the  $d$ -dimensional situation by one-to-one mapping between  $\mathbb{N}^d$  and  $\mathbb{N}$ ; it is based on the Hausdorff inequality [12, th. IX.8, p.11], which is itself a consequence of the Riesz–Thorin interpolation theorem [12, th. IX.17 p.27]. These theorems also give a lower bound for the product of the entropy powers  $N_{\frac{p}{2}}(A)N_{\frac{q}{2}}(\tilde{A})$  in the continuous-continuous case, but this bound is not sharp. However, this case is addressed via the Beckner relation given in Ref. [13], as shown in [1,2,14]. Finally, we remark that the product  $N_{\frac{p}{2}}(A)N_{\frac{q}{2}}(\tilde{A})$  is scale invariant in the sense that it is independent of invertible matrix  $M$  when replacing  $\Psi(x)$  by  $|M|^{-\frac{d}{2}}\Psi(M^{-1}x)$ .

It is worth stressing that inequality (7) provides a universal (*i.e.*, independent of the state of the system) lower bound for the product of two measures of uncertainty that quantify the missing information related with the measurement or preparation of the system with operators  $A$  and  $\tilde{A}$ . We note that in the continuous-continuous context, equality is reached if and only if  $\Psi$  is a Gaussian wave function; in the case of discrete states, equality is attained if and only if  $\Psi$  coincides with a Kronecker indicator, or with a constant (by conjugation, and provided that the space state is finite). Moreover, maximization of  $(H_1(A) + H_1(\tilde{A}))/d$  ( $p/2 = q/2 = 1$  conjugated) without constraint has been suggested as an interesting counterpart to the usual approach of entropy maximization under constraint for the derivation of the wavefunction associated to atomic systems [15].

It is important to note that all uncertainty relations (7) deal with Rényi entropies with *conjugated* indices. Indeed, the Hausdorff inequality, as a consequence of the Parseval relation expressing the conservation by Fourier trans-

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<sup>2</sup> Rigorously speaking,  $p$  and  $q$  are conjugated if  $1/p + 1/q = 1$ . Thus here,  $p/2$  and  $q/2$  are not conjugated; however, in the rest of the paper, and for the sake of simplicity, we will call  $p/2$  and  $q/2$  such that  $1/p + 1/q = 1$  “conjugated” indices.

formation of the Euclidean norm, and from the trivial relation  $\|\Psi\|_\infty \leq \|\Psi\|_1$ , involves only conjugated indices, as recalled *e.g.* in Ref. [8]. In the following section, we study extensions of these uncertainty inequalities to the case of arbitrary pairs of *non-conjugated* indices. Such a situation has been considered by Luis in [3] for a particular combination of indices to describe the uncertainty product of exponential states. Moreover, one may hope to gain more flexibility by using non-conjugated information measures in both spatial and momentum domains. As a supplementary motivation, one may wish to quantify uncertainty for conjugate observables using the same entropic measure  $N_\lambda$ : inequality (7) then holds only when  $p = q = 2$  (Shannon case).

### 3 Uncertainty relations with entropies of arbitrary indices

We present here an extension of entropic formulations of the Uncertainty Principle for conjugate observables, to the general situation of arbitrary pairs of entropic indices. For this purpose, we discuss separately the three different cases corresponding to operators having discrete or continuous spectrum.

#### 3.1 Discrete–discrete case

Inequality (7) appears in Ref. [8] in the context of finite (discrete) states and  $d = 1$ , but an extension to arbitrary nonnegative indices  $\alpha$  and  $\beta$  can also be provided [8,16], which reads with the notation adopted here

$$N_\alpha(A)N_\beta(\tilde{A}) \geq \left(\frac{2n}{n+1}\right)^2 \quad (9)$$

This uncertainty relation is first proved in the case  $\alpha$  and  $\beta$  infinite and then, for any pair of indices  $\alpha$  and  $\beta$  as a consequence of the positivity and the non-increasing property of  $N_\lambda$  in  $\lambda$  (see above and [9, th. 16]). This inequality was rediscovered recently by Luis in the particular case of non-conjugated indices  $\alpha = \beta = 2$  [3, eq. 2.8]. Again, equality is reached in (9) when  $\Psi$  coincides with a Kronecker indicator and  $\hat{\Psi}$  is constant, or “conjugately”, for a constant  $\Psi$ .

#### 3.2 Continuous–continuous case

##### 3.2.1 A result by Luis

In Ref. [3], Luis claims that, to his best knowledge, there is no known continuous counterpart of the finite discrete case. Furthermore, after mentioning

that only Gaussian wavefunctions saturate both the Heisenberg uncertainty relation and the conjugated-indices entropic uncertainty relation (7), Luis considers the product of entropy powers in the particular case  $\alpha = \beta = 2$ . Indeed, he compares this product, for some particularly distributed states, to the value obtained for Gaussian states  $\Psi_G$  associated to operator  $G$ . He remarks that this product is not saturated by Gaussians since, for one-dimensional exponential states  $\Psi_E$  (resp. operator  $E$ ), he finds that

$$N_2(E)N_2(\tilde{E}) = \frac{8\pi}{5} < 2\pi = N_2(G)N_2(\tilde{G}). \quad (10)$$

In what follows, we explain this result and deduce extended versions of the entropic uncertainty relations for arbitrary indices.

### 3.2.2 Uncertainty relations for arbitrary indices and domain of existence

For any real  $\alpha \geq \frac{1}{2}$ , let us introduce the notation

$$\tilde{\alpha} = \frac{\alpha}{2\alpha - 1} \quad (11)$$

so that  $\alpha$  and  $\tilde{\alpha}$  are conjugated indices, *i.e.*  $1/\alpha + 1/\tilde{\alpha} = 2$  (for  $\alpha = 1/2$ ,  $\tilde{\alpha}$  is infinite), and the function  $B : [1/2 ; +\infty) \mapsto [2\pi ; e\pi]$

$$B(\alpha) = \pi \alpha^{\frac{1}{2(\alpha-1)}} \tilde{\alpha}^{\frac{1}{2(\tilde{\alpha}-1)}} \quad (12)$$

When  $\alpha \rightarrow \frac{1}{2}^+$ ,  $B(\alpha) \rightarrow 2\pi$ ; hence, by continuity, we set  $B(1/2) = 2\pi$ . By continuity, we also set  $B(1) = e\pi$ . We also define the following sets on the plane

$$\left\{ \begin{array}{l} \mathcal{D}_0 = \left\{ (\alpha, \beta) : \alpha > \frac{1}{2} \text{ and } \beta > \tilde{\alpha} \right\} \\ \mathcal{S} = \left[ 0 ; \frac{1}{2} \right)^2 \\ \mathcal{D} = \mathbb{R}_+^2 \setminus \mathcal{D}_0 \\ \mathcal{C} = \left\{ (\alpha, \beta) : \alpha \geq \frac{1}{2}, \beta = \tilde{\alpha} \right\}. \end{array} \right. \quad (13)$$

which are represented in Fig. 2. The solid line represents curve  $\mathcal{C}$  for which  $\alpha$  and  $\beta$  are conjugated and the shaded region represents  $\mathcal{D}$ .

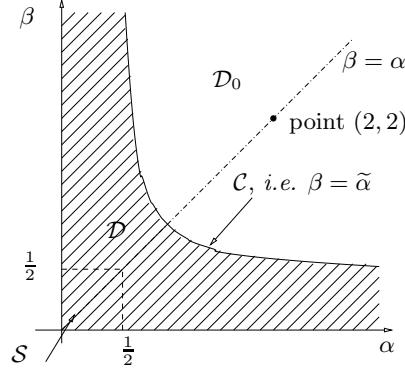


Fig. 2. Sets  $\mathcal{D}_0$  (blank region),  $\mathcal{D}$  (shaded region),  $\mathcal{S}$  (square in the shaded region) and  $\mathcal{C}$  (solid line) in the  $(\alpha, \beta)$ -plane, as given by Eq. (13). Notice that, by definition,  $\mathcal{C} \subset \mathcal{D}$

We recall that most of the previous results on uncertainty relations in terms on  $\lambda$ -Rényi entropies refer to inequalities for pairs of conjugated entropic indices *i.e.* located on curve  $\mathcal{C}$  (see Eqs. (7)-(8) above). We first remark that, as for conjugated indices, the product  $N_\alpha(A)N_\beta(\tilde{A})$  is scale invariant for any pair of indices  $(\alpha, \beta) \in \mathbb{R}_+^2$ . We now introduce some novel results for *arbitrary pairs of indices*, depending on the region of  $\mathbb{R}_+^2$  where they lie.

**Result 1** *For any pair  $(\alpha, \beta) \in \mathcal{D}$  and for conjugate operators  $A$  and  $\tilde{A}$ , there exists an uncertainty principle under the form*

$$N_\alpha(A)N_\beta(\tilde{A}) \geq B_{\alpha, \beta}, \quad (14)$$

where

$$B_{\alpha, \beta} = \begin{cases} B(\alpha) & \text{in } (\mathcal{D} \setminus \mathcal{S}) \cap \{(\alpha, \beta) : \alpha \geq \beta\} \\ B(\beta) & \text{in } (\mathcal{D} \setminus \mathcal{S}) \cap \{(\alpha, \beta) : \beta \geq \alpha\} \\ 2\pi & \text{in } \mathcal{S}. \end{cases} \quad (15)$$

The proof of this result is given in appendix A. Except on  $\mathcal{C}$ , the bound  $B_{\alpha, \beta}$  is probably not sharp and we have not determined if such an uncertainty saturates for Gaussians or not. A direct calculation with Gaussians shows that  $N_\alpha(G)N_\beta(\tilde{G}) = \pi\alpha^{\frac{1}{2(\alpha-1)}}\beta^{\frac{1}{2(\beta-1)}}$  which is strictly higher than  $B_{\alpha, \beta}$  in  $\mathcal{D} \setminus \mathcal{C}$ : thus, either bound (15) is not sharp in  $\mathcal{D} \setminus \mathcal{C}$ , or Gaussians do not saturate (14)–(15) in  $\mathcal{D} \setminus \mathcal{C}$ , or both. This point remains to be solved. Note however that the point  $(0, 0)$  is degenerate since  $N_0(A)$  measures the volume of the support of  $A$ . It is well known that if  $\Psi$  is defined on a finite volume support, the support of its Fourier transform has infinite measure. Hence  $N_0(A)N_0(\tilde{A}) = +\infty$  which

trivially fulfills (14)-(15). By continuity of  $N_\alpha$  in  $\alpha$ , this remark suggests that the bound (15) is not sharp, at least in  $\mathcal{S}$ .

Our second result concerns the non-existence of a generalized entropy formulation for the uncertainty principle in the area  $\mathcal{D}_0$ .

**Result 2** *For any pair  $(\alpha, \beta) \in \mathcal{D}_0$  and for conjugate operators  $A$  and  $\tilde{A}$ , the positive product of the entropy powers  $N_\alpha(A)N_\beta(\tilde{A})$  can be arbitrarily small. In other words, no entropic uncertainty principle exists in  $\mathcal{D}_0$ .*

The proof and illustrations of this result are given in appendix B. Figure 3 schematizes the preceding results: entropic uncertainty relations exist in  $\mathcal{D}$  but not in  $\mathcal{D}_0$ .

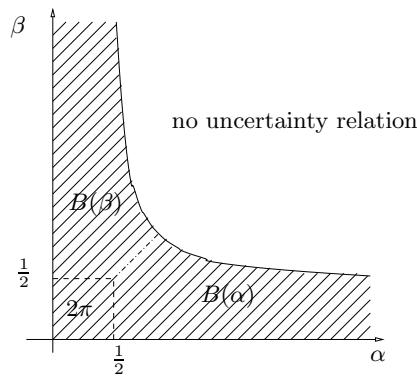


Fig. 3. Entropic uncertainty relations exist for  $(\alpha, \beta) \in \mathcal{D}$  as depicted by the shaded area, with the corresponding bound. Conversely, in  $\mathcal{D}_0$  the positive product  $N_\alpha(A)N_\beta(\tilde{A})$  can be arbitrarily small.

In the case studied by Luis [3],  $(\alpha, \beta) = (2, 2) \in \mathcal{D}_0$  (see Fig. 2) so that one can find wave functions (or operators) for which  $N_2(A)N_2(\tilde{A})$  can be arbitrarily small. This is thus not surprising that Luis finds a wavefunction for which this entropy power is lower than in the Gaussian case: he considers in fact the special one dimensional Student-t case with  $\nu = 3$  of Eq. (B.2) below; varying parameter  $\nu$ , one can describe a richer family of distributions that allows to prove result 2. Other cases having equal entropic indices are worth studying in more details. These cases correspond to points located along the line that bisects  $\mathbb{R}_+^2$  (see Fig. 2) and, as mentioned, the situation is very different whether the point lies inside  $\mathcal{D}$  or inside  $\mathcal{D}_0$ .

Figure 4 depicts the behavior of the product  $N_\alpha(A)N_\alpha(\tilde{A})$  versus  $\alpha > 0$ , in the Gaussian context, in the Student- $t$ /Laplace context of Luis, and in a specific Student- $t$  case used in the proof of result 2. This illustration is motivated by the interest in using the same entropic measure to describe uncertainty for conjugate observables. It clearly seen that, when  $\alpha > 1$ , no uncertainty principle holds. Moreover, for many operators  $A$ ,  $N_\alpha(A)N_\alpha(\tilde{A})$  can be below the product for the Gaussian case  $N_\alpha(G)N_\alpha(\tilde{G})$ , for a wide

range of index  $\alpha$ .

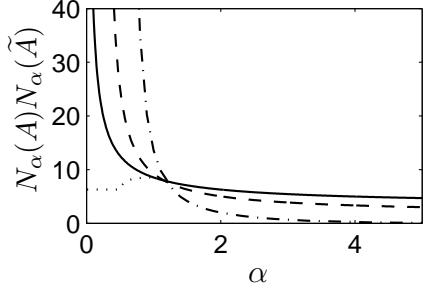


Fig. 4. Behavior of  $N_\alpha(A)N_\alpha(\tilde{A})$  versus  $\alpha$  for various wavefunctions ( $d = 1$ ) : Gaussian case (solid line), Student- $t$  case  $\Psi(\mathbf{x}) \propto (1 + \mathbf{x}^t \mathbf{x}/(\nu - 2))^{-\frac{d+\nu}{4}}$  for  $\nu = 3$  (dashed line) and Student- $t$  case for  $\nu = .8$  (dash-dotted line). The dotted line depicts the lower bounds (15) that only exist when  $\alpha \leq 1$ .

Concerning our second result for any pair of Rényi indices  $(\alpha, \beta) \in \mathcal{D}_0$  and conjugate operators  $A$  and  $\tilde{A}$ , two alternative interpretations arise: either uncertainty principle does not apply in those cases, which contradicts physical intuition, or the Rényi entropy-power product is not suitable to quantify uncertainty for conjugate observables in the area  $\mathcal{D}_0$ . The Rényi as well as Shannon entropy related to a given observable quantifies the amount of missing information in the knowledge of the properties of a system in connection with a measurement of that observable. It is in this sense that one can consider  $H_\lambda$  as a measure of *uncertainty*. In order not to contradict the common principle of uncertainty in quantum physics for pairs of non-commuting observables, one can argue that for  $\alpha$  fixed and a given quantum observable, the  $\beta$ -Rényi entropy, when  $\beta$  is such that  $(\alpha, \beta) \in \mathcal{D}_0$ , is not well suited to describe the lack of knowledge about the conjugate observable. It is interesting to note that for  $\alpha = \beta = 2$ , the entropy power product is the Onicescu measure used to quantify complexity or disequilibrium in atomic systems[17,18]. However, it has been suggested to use this product  $N_2(A)N_2(\tilde{A})$ , divided by the product of the Fisher informations related to  $A$  and  $\tilde{A}$ . This can suggest that in itself,  $N_2(A)N_2(\tilde{A})$  is not a reasonable measure to account for uncertainty for conjugate observables.

### 3.3 Discrete–continuous case

Our last result concerns the discrete–continuous case, and can be expressed as follows.

**Result 3** *For any pair  $(\alpha, \beta) \in \mathcal{D}$  and for conjugate  $A$  and  $\tilde{A}$ , there exists an uncertainty principle under the form*

$$N_\alpha(A)N_\beta(\tilde{A}) \geq 2\pi. \quad (16)$$

The proof is similar to that of result 1. The bound comes from (8). We remark that here, the lower bound of  $N_\alpha(A)N_\beta(\tilde{A})$  is sharp and is attained when  $\Psi(\mathbf{k})$  coincides with a Kronecker indicator.

What happens in the  $\mathcal{D}_0$  domain remains to be solved.

## 4 Conclusions

We have addressed some fundamental questions related with the formulation of the Uncertainty Principle for pairs of conjugate operators, like *e.g.* position and momentum, in entropic terms. Our study extends the set of uncertainty inequalities as derived by Bialynicki-Birula [1] to the case where the entropic indices are not conjugated. Our main findings are summarized in results 1, 2 and 3 that establish the conditions under which an entropic formulation of the Uncertainty Principle makes sense and, if so, its lower bound. We have addressed the cases where both state space and Fourier transformed state space are respectively (i) discrete and discrete, (ii) continuous and continuous, and (iii) discrete and continuous. The cases of equality in the uncertainty relation considered (*i.e.*, the state of the system corresponding to minimum uncertainty in the simultaneous measurement of both observables) are still undetermined and will be the object of further research.

Summing up, our study establishes very general conditions for the formulation of the Uncertainty Principle in entropic terms (as an alternative to the Robertson–Schrödinger formulation in terms of variances), making use of generalized entropies as measures of uncertainty for the preparation or measurement of pairs of quantum observables in a given state of a physical system. We believe that our analysis sheds some light on previous related work in the field, and also that it has implications in the discussion of quantum behavior of physical systems, like quantum limits to precision measurements, for instance.

## A Proof of Result 1

One deals here with entropic uncertainty products of the form  $N_\alpha(A)N_\beta(\tilde{A})$  for arbitrary pairs of indices  $(\alpha, \beta) \in \mathcal{D}$  and for conjugate operators  $A$  and  $\tilde{A}$  having continuous spectra. We first prove existence and then evaluate lower bounds in different regions inside set  $\mathcal{D}$ .

- (1) From (7)–(8), setting  $p = 2\alpha$  (then  $q = 2\tilde{\alpha}$ ), one has  $N_\alpha(A)N_{\tilde{\alpha}}(\tilde{A}) \geq B(\alpha)$ . This proves the result for indices on curve  $\mathcal{C}$ .
- (2) In order to prove the result for indices in  $\mathcal{D} \setminus \mathcal{C}$  we first restrict the proof to  $\mathcal{D} \setminus \mathcal{S}$  in two steps:
  - (a) Fix  $\alpha \geq \frac{1}{2}$ : since the Rényi entropy power is decreasing in  $\beta$  [9, th. 192], for all  $\beta \leq \tilde{\alpha}$ , inequality  $N_\alpha(A)N_\beta(\tilde{A}) \geq B(\alpha)$  still holds. This case is schematized in Fig. A.1(a).

(b) The same arguments apply by symmetry between  $\alpha$  and  $\beta$ :  $N_\alpha(A)N_\beta(\tilde{A}) \geq B(\beta)$  provided that  $\alpha \leq \tilde{\beta}$ . This case is schematized in Fig. A.1(b). As a conclusion, uncertainty relation (14) exists provided  $\alpha \geq 1/2$  and  $\beta \leq \tilde{\alpha}$  (first case), or provided  $\beta \geq 1/2$  and  $\alpha \leq \tilde{\beta}$  i.e. provided  $(\alpha, \beta) \in \mathcal{D} \setminus \mathcal{S}$ .

(3) Let us consider now the set  $\mathcal{S}$ : uncertainty relation (14) remains true in the limit  $\alpha \rightarrow 1/2$ , and hence in point  $(1/2; 1/2)$ . Thus, with the same argument of non-increasing property of  $N_\alpha$ , an uncertainty relation again exists in  $\mathcal{S}$ ,  $N_\alpha(A)N_\beta(\tilde{A}) \geq N_{1/2}(A)N_{1/2}(\tilde{A}) \geq N_{1/2}(A)N_\infty(\tilde{A})$ .

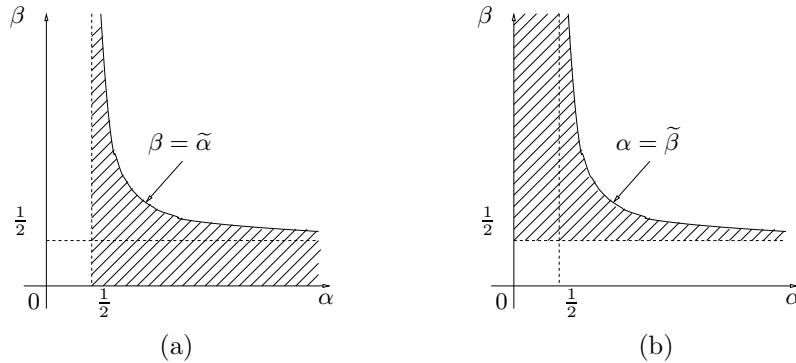


Fig. A.1. (a) The dashed area represents the area where for any fixed  $\alpha$  and for  $\tilde{\alpha} \geq \beta \geq 0$ , uncertainty relation (7) extends to. (b): The problem is symmetric by exchanging  $\alpha$  and  $\beta$ ; the dashed area represents then the “conjugated” area dashed in (a).

As a conclusion, an uncertainty principle exists in the whole area  $\mathcal{D}$ .

In order to evaluate the bound  $B_{\alpha,\beta}$ , we restrict our attention to the case  $\alpha \geq \beta$  since trivially  $B_{\alpha,\beta} = B_{\beta,\alpha}$ .

- For  $\alpha \geq \frac{1}{2}$  and  $\beta < \frac{1}{2}$ , from (8) or figure A.1, only case (2a) is to be considered. Hence,  $B_{\alpha,\beta} = B(\alpha)$ .
- For  $\alpha \in (\frac{1}{2}; 1]$ , a study of function  $B(\alpha)$  shows that it increases in  $(\frac{1}{2}; 1]$  from value  $2\pi$  to  $\pi e$ . In the situation  $\alpha \in (\frac{1}{2}; 1]$  and  $\beta \geq \frac{1}{2}$ , both cases (2a) and (2b) occur (see figure A.1) but since  $\beta \leq \alpha$ ,  $\max(B(\alpha), B(\beta)) = B(\alpha)$ . Thus, again  $B_{\alpha,\beta} = B(\alpha)$ .
- When  $\alpha > 1$ , the study of  $B$  shows also that it decreases in  $(1; +\infty)$  (from  $\pi e$  to  $2\pi$ ) and furthermore that for  $\beta \leq \tilde{\alpha}$ ,  $B(\beta) \leq B(\alpha)$ . Here again  $\max(B(\alpha), B(\beta)) = B(\alpha)$  and thus  $B_{\alpha,\beta} = B(\alpha)$ .
- Finally, when  $\alpha < 1/2$ , from case 3 and  $B\left(\frac{1}{2}\right) = 2\pi$ , one has  $B_{\alpha,\beta} = 2\pi$  in  $\mathcal{S}$ .

## B Proof of Result 2

To prove this result, it is sufficient to exhibit an example for which  $N_\alpha(A)N_\beta(\tilde{A})$  can be arbitrarily small when  $(\alpha, \beta) \in \mathcal{D}_0$ . To this aim, let us consider the following Student- $t$  wavefunction

$$\Psi(\mathbf{x}) = \sqrt{\frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{\nu}{2}\right)}} \left(1 + \mathbf{x}^t \mathbf{x}\right)^{-\frac{d+\nu}{4}}, \quad (\text{B.1})$$

where  $\Gamma$  is the Gamma function and  $\nu > 0$  is a parameter called degree of freedom. Its Fourier transform, from Refs. [19, eq. 5] and [20, 6.565-4], reads

$$\hat{\Psi}(\mathbf{x}) = \sqrt{\frac{2^{\frac{4-d-\nu}{2}}\Gamma\left(\frac{d+\nu}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{d+\nu}{4}\right)}} (\mathbf{x}^t \mathbf{x})^{\frac{\nu-d}{8}} K_{\frac{d-\nu}{4}}((\mathbf{x}^t \mathbf{x})^{\frac{1}{2}}), \quad (\text{B.2})$$

where  $K_\mu$  is the modified Bessel function of the second kind of order  $\mu$ . From Ref. [20, 4.642 and 8.380-3], the Rényi  $\alpha$ -entropy power associated to  $\Psi$  is expressed as

$$N_\alpha(A) = \sqrt{\pi} \left( \frac{\Gamma^\alpha\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{\alpha(d+\nu)}{2}\right)} \right)^{\frac{1}{d(1-\alpha)}} \left( \frac{\Gamma\left(\frac{\alpha(d+\nu)-d}{2}\right)}{\Gamma^\alpha\left(\frac{\nu}{2}\right)} \right)^{\frac{1}{d(1-\alpha)}} \quad \text{if } \alpha \neq 1. \quad (\text{B.3})$$

The case  $\alpha = 1$  is obtained by continuity. Rényi  $\alpha$ -entropy power is then defined provided that  $\nu > \max(0, \frac{d(1-\alpha)}{\alpha})$ , *i.e.*  $\nu > 0$  if  $\alpha > 1$  and  $\nu > \frac{d(1-\alpha)}{\alpha}$  otherwise.

Similarly, the Rényi  $\beta$ -entropy power associated to  $\hat{\Psi}$  is expressed as

$$N_\beta(\tilde{A}) = \sqrt{\pi} \left( \frac{2^{\frac{(4-d-\nu)\beta+2}{2d}}\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma^2\left(\frac{d+\nu}{4}\right)} \right)^{\frac{\beta}{d(1-\beta)}} \times \left( \frac{\int_0^{+\infty} r^{d-1+\frac{\beta(\nu-d)}{2}} K_{\frac{d-\nu}{4}}^{2\beta}(r) dr}{\Gamma^\beta\left(\frac{\nu}{2}\right)} \right)^{\frac{1}{d(1-\beta)}} \quad \text{if } \beta \neq 1, \quad (\text{B.4})$$

the case  $\beta = 1$  being obtained by continuity. From the properties of the Gamma function, from [21, 9.6.8 and 9.6.9] and since  $K_\mu = K_{-\mu}$ , the Rényi

$\beta$ -entropy power exists provided that  $\nu > \max(0, \frac{d(\beta-1)}{\beta})$ , i.e.  $\nu > 0$  if  $\beta < 1$  and  $\nu > \frac{d(\beta-1)}{\beta}$  otherwise.

From the domain of existence of  $N_\beta(\tilde{A})$ , we will distinguish the cases  $\beta > 1$ ,  $\frac{1}{2} \leq \beta < 1$  and the limit case  $\beta = 1$ . Playing with the degree of freedom  $\nu$ , we will show that  $N_\alpha(A)N_\beta(\tilde{A})$  can be arbitrarily small.

- Consider first the case  $\beta > 1$ . Then fix  $(\alpha, \beta) \in \mathcal{D}_0$  and consider further the case where  $\frac{d(\beta-1)}{\beta} < \nu \leq d$ . One can easily check that  $\frac{\alpha(d+\nu)-d}{2} > \frac{d}{2} \left( \frac{\alpha}{\beta} - 1 \right)$ : from (B.3)  $N_\alpha(A)$  exists and is finite whatever  $\nu \in \left[ \frac{d(\beta-1)}{\beta}, d \right]$ . Moreover, from the integral term of (B.4) and [21, 9.6.9], one can check that  $\lim_{\nu \rightarrow \frac{d(\beta-1)}{\beta}} N_\beta(\tilde{A}) = 0$ . As a consequence, for any  $(\alpha, \beta) \in \mathcal{D}_0 \cap \{(\alpha, \beta) | \beta > 1\}$ , we have  $\lim_{\nu \rightarrow \frac{d(\beta-1)}{\beta}} N_\alpha(A)N_\beta(\tilde{A}) = 0$ , which proves that the product  $N_\alpha(A)N_\beta(\tilde{A})$  can be arbitrarily small in  $\mathcal{D}_0 \cap \{(\alpha, \beta) | \beta > 1\}$ .
- consider now the case  $\frac{1}{2} \leq \beta < 1$ , and  $0 < \nu$ . Then fix again a pair  $(\alpha, \beta) \in \mathcal{D}_0$ . One can check that  $\frac{\alpha(d+\nu)-d}{2} > \frac{d}{2}(\alpha - 1) > 0$ . The last inequality comes from  $(\alpha, \beta) \in \mathcal{D}_0$ , in which if  $\beta < 1$  then  $\alpha > 1$ . Hence, from (B.3)-(B.4) one can write

$$N_\alpha(A)N_\beta(\tilde{A}) \propto \Gamma^{\frac{\alpha}{d(\alpha-1)} + \frac{\beta}{d(\beta-1)}} \left( \frac{\nu}{2} \right)$$

where the coefficient of proportionality exists and is finite for any  $\nu \geq 0$ . Since in  $\mathcal{D}_0$  one has  $\frac{\alpha}{\alpha-1} + \frac{\beta}{\beta-1} < 0$ , we obtain that  $\lim_{\nu \rightarrow 0} N_\alpha(A)N_\beta(\tilde{A}) = 0$ , which proves that the product  $N_\alpha(A)N_\beta(\tilde{A})$  can be arbitrarily small in  $\mathcal{D}_0 \cap \{(\alpha, \beta) | \beta < 1\}$ .

- The case  $\beta = 1$  can be deduced by continuity of  $N_\beta$  as a function of  $\beta$ .

As a conclusion, this example shows that for any pair of entropic indices in  $\mathcal{D}_0$ , the positive quantity  $N_\alpha(A)N_\beta(\tilde{A})$  can be arbitrarily small. This is illustrated on figures B.1 where the behavior of  $N_\alpha(A)N_\beta(\tilde{A})$  versus  $\nu$  in the Student- $t$  case is depicted, for chosen pairs  $(\alpha, \beta) \in \mathcal{D}_0$ . For the sake of comparison, let us recall that a non-trivial bound for the power-entropies product exists in the case of pairs of entropic indices located in  $\mathcal{D}$ , as given by Result 14. This is illustrated in Fig. B.2, where the behavior of  $N_\alpha(A)N_\beta(\tilde{A})$  versus  $\nu$  in the Student- $t$  case is depicted, for chosen pairs  $(\alpha, \beta) \in \mathcal{D}$ .

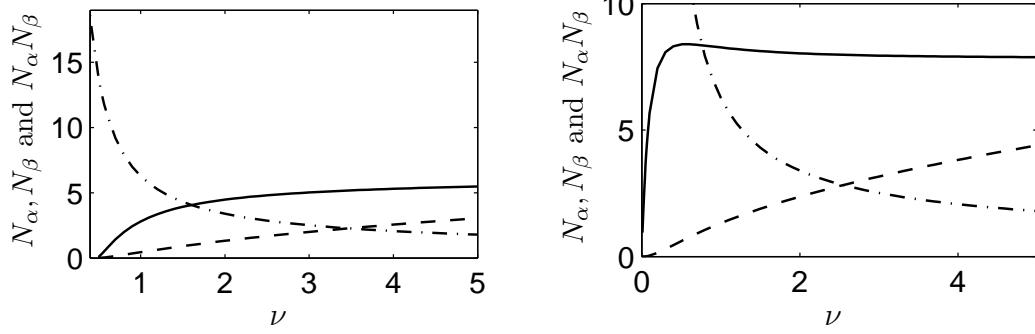


Fig. B.1. Behavior of  $N_\alpha(A)$  (dash-dotted line),  $N_\beta(\tilde{A})$  (dashed line), and of the product  $N_\alpha(A)N_\beta(\tilde{A})$  (solid line) versus  $\nu$  in the Student- $t$  case described in the proof of result 2 (see text). In these illustrations  $d = 1$ ,  $(\alpha, \beta) = (2, 2)$  (left) and  $(\alpha, \beta) = (2, 3/4)$  (right) are both in  $\mathcal{D}_0$ . It confirm that, when  $\beta > 1$  we have  $\lim_{\nu \rightarrow \frac{d(\beta-1)}{\beta}} N_\beta(\tilde{A}) = 0$  while  $N_\alpha(A)$  remains finite for any  $\nu > 0$ :  $N_\alpha(A)N_\beta(\tilde{A})$  can be arbitrarily small. Likewise, when  $\beta < 1$ ,  $\lim_{\nu \rightarrow 0} N_\beta(\tilde{A}) = 0$  and  $\lim_{\nu \rightarrow 0} N_\alpha(A) = +\infty$ , but  $\lim_{\nu \rightarrow 0} N_\alpha(A)N_\beta(\tilde{A}) = 0$ .

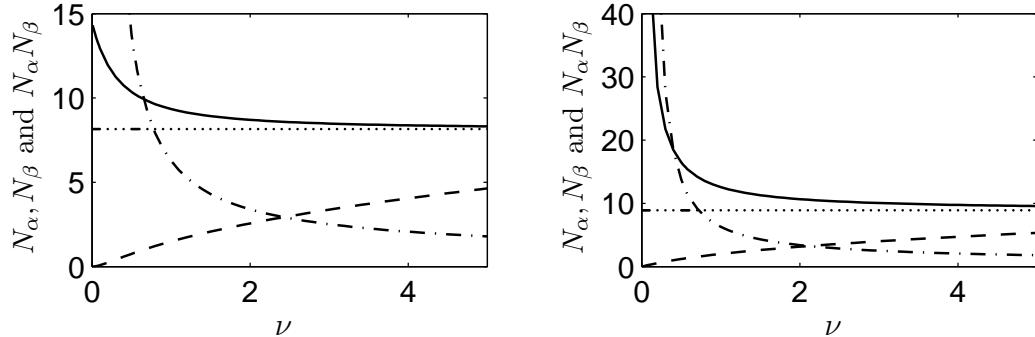


Fig. B.2. Behavior of  $N_\alpha(A)$  (dash-dotted line),  $N_\beta(\tilde{A})$  (dashed line), and of the product  $N_\alpha(A)N_\beta(\tilde{A})$  (solid line) versus  $\nu$  in the Student- $t$  case, to illustrate result (14) (see text). Here  $(\alpha, \beta) = (2, 2/3) \in \mathcal{C}$  (left) and  $(\alpha, \beta) = (2, 1/2) \in \mathcal{D} \setminus \mathcal{C}$  (right). In both cases one has,  $\lim_{\nu \rightarrow 0} N_\beta(\tilde{A}) = 0$  and  $\lim_{\nu \rightarrow 0} N_\alpha(A) = +\infty$ , but on  $\mathcal{C}$  the product  $N_\alpha(A)N_\beta(\tilde{A})$  has a finite limit while on  $\mathcal{D} \setminus \mathcal{C}$  the limit is infinite. The dotted line represents bound (15) corresponding to inequality (14).

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